8.2: Discrete Probability Model

We will work with two kinds of probability models. The first is the discrete probability model.

Discrete probability model: (discrete = finite space, countable number of outcomes)

- In discrete probability models, you are able to individually list out all events in the sample space and assign probabilities to each event.
- This is done by:
    - Listing the probability of all the individual outcomes
        - Rule 1: these probabilities must be numbers between 0 and 1 inclusive and must have sum of 1.
        - Rule 2: probability of any event is the sum of the probability of the outcomes making up the event

- In discrete probability models, <u>**probability histograms**</u> graphically show the likelihood of each outcome.
    - Height of each bar shows the probability of the outcomes at its bases, and the sum of the height is 1.
    - Probability histograms are symmetrical

8.3: Finding Probabilities of Equally Likely Outcomes

If a random phenomenon has n possible outcomes, all <u>equally likely</u> then each individual outcome has a probability of 1/n.

The probability of any event I is as follows:

$$P(A) = \frac{\text{count of outcomes in event A}}{\text{count of outcomes in sample space}} = \frac{\text{count of outcomes in event A}}{\text{n}}$$

When outcomes are equally likely, we find probabilities by counting outcomes.

<u>**Combinatorics**</u>: the study of methods of counting

Using the Fundamental Principle of Counting, one can get a handle on some basic counting problems.

<u>**Fundamental Principle of Counting**</u>: If there are *a* ways of choosing one thing, *b* ways of choosing a second after the first is chosen, . . . , and *z* ways of choosing the last item after the earlier choices, then the total number of choices sequences is *a* X *b* X . . . X *z*.

- Rule A: Arranging *k* objects chosen from a set of *n* distinct possibilities, with **repetitions allowed**, can be done in $n \times n \times \ldots \times n = n^k$ distinct ways

Example: A strand of DNA is a long sequence of the nucleotides adenine, cytosine, guanine, and thymine (abbreviated A, C, G, T). One helical turn of a DNA strand would contain a sequence of 10 of these acids, such as ACTGCCATGT. How many possible sequences of this length are there?

- There are 4 letters that can occur in each position in the 10-letter sequence.
- Any of the 4 letters can be in the first position.
- Regardless of what is in the fist position, any of the 4-letters can be in the second position, and so on.
- The order of the letter matters, so a sequence of that begins with AC will be a different sequence than one that begins with CA

- 4 X 4 X 4 X 4 X 4 X 4 X 4 X 4 X 4 X 4 = 4^{10} = 1,048,576 is the number of different 10-letter sequences

Example: A Major League Baseball team has 25 players on the active roster who are eligible to play in a game. At the start of the game, the manager gives the officiating crew a list of the team's 9 hitters who will begin the game and in what order they will bat.

- Order matters
- Listing the same item more than once is not allowed
- Any of the 25 players can be chose to bat first, but only the remaining 24 players are available to be listed as the second batter, so that there are 25 X 24 choices for the first two batters
- Any of these choices leaves 23 batters for the first position, and so on.

- 25 X 24 X 23 X 22 X 21 X 20 X 19 X 18 X 17 = 741,354,768,000 is the number of different batting lineups

This baseball lineup scenario--choosing an ordered subset of *k* players from a roster of *n* players—is called a **permutation**.

Permutation: A **permutation** is an ordered arrangement of *k* items that are chosen without replacement from a collection of *n* items. It can be notated as P(n, k) or $_nP_k$.

$$_nP_k = \frac{n!}{(n-k)!}$$

Factorial: (n!) equals the product of the first *n* positive integers.

$n!$ = n factorial.

- $5! = 5 \times 4 \times 3 \times 2 \times 1$
- $\frac{7!}{5!} = \frac{7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{5 \times 4 \times 3 \times 2 \times 1}$
- $\frac{10!}{3!7!} = \frac{10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{3 \times 2 \times 1 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}$

- Rule B (Permutations): Arranging *k* objects chosen from a set of *n* distinct possibilities, with **no repetitions allowed**, can be done in $n \times (n-1) \times \ldots \times (n-k+1)$. (Notice that they are *k* factors here). This is called a **permutation**. $_nP_k = \frac{n!}{(n-k)!}$

Using the baseball example:

$$_{25}P_9 = \frac{25!}{(25-9)!} = \frac{25!}{16!} = 25 \cdot 24 \cdot 23 \cdot 22 \cdot 21 \cdot 20 \cdot 19 \cdot 18 \cdot 17$$

Example: Suppose you have 5 different books in wish to arrange 3 on a shelf. How many different arrangements are there? Order does matter.

- Since order matters, we will calculate

$${}_5P_3 = \frac{5!}{(5-3)!} = \frac{5!}{2!} = \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{2 \cdot 1} = 5 \cdot 4 \cdot 3 = 60$$ different arrangements

Combination: Order doesn't matter (combination lock is inaccurately named b/c order does matter)
Permutation: Order does matter

Winning the Lottery.
In a typical state or multi-state lottery game you win (at least a share of) the jackpot as long as the collection of numbers you pick is the same collection that the Lottery selects. This particular lottery requires choosing the right set of 6 distinct (different) balls out of a collection of 46 numbers.

- Repetition is not allowed: the same number can't be picked twice in the same drawing
- Unlike permutations, order does not matter here. It doesn't matter which order the numbered ping pong balls come out of the mixing chamber—all that matters is what numbers are selected to be in the that drawing's group of winners.
- We will use a modification of the permutation approach
- The number of ordered sets will be much larger than the number of unordered sets since the lottery drawing {2, 14, 15, 21, 30, 33} is the same set of balls as {15, 2, 30, 14, 33, 21} for example.
- So the number of collections of lottery balls will simply be the number of permutations divided by k!.

$${}_{46}P_6 = \frac{46!}{6!(46-6)!} = \frac{46!}{6!(40)!} = \frac{46 \cdot 45 \cdot 44 \cdot 43 \cdot 42 \cdot 41}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 9{,}366{,}819$$
possible set of numbers

* the probability of holding a winning ticket is: $\frac{1}{9{,}366{,}819}$

Example: Suppose you have 7 distinct books and wish to collect 5 of them to donate to a school. In how many ways can this be done?

- Since the order doesn't matter, we seek ${}_7C_5$.

$${}_7C_5 = \frac{7!}{5!(7-5)!} = \frac{7!}{5!2!} = \frac{7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}{(5\cdot 4\cdot 3\cdot 2\cdot 1)\cdot(2\cdot 1)} = \frac{7\cdot 6}{2\cdot 1} = 7\cdot 3 = 21$$

<u>Rules</u>:

- Rule A: Arranging *k* objects chosen from a set of *n* distinct possibilities, with **repetitions allowed**, can be done in $n \times n \times \ldots \times n = n^k$ distinct ways
- Rule B: (Permutations) Arranging *k* objects chosen from a set of *n* distinct possibilities, with **no repetitions allowed**, can be done in $n \times (n-1) \times \ldots \times (n-k+1)$. (Notice that they are *k* factors here). This is called a **permutation**. ${}_nP_k = \frac{n!}{(n-k)!}$
- Rule C: Collecting (unordered arrangement) k objects chosen from a set of *n* distinct possibilities, with **repetitions allowed**, can be done in ${}_{n+k-1}C_k = \frac{(n+k-1)!}{k!(n-1)!}$ distinct ways.
- Rule D: Collecting (unordered arrangement) *k* objects chosen from a set of *n* distinct possibilities, with **no repetitions allowed**, can be done in ${}_nC_k = \frac{n!}{k!(n-k)!}$ ways.